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SUPERSYMMETRIC YANG-MIILS QUANTUM MECHANICS IN VARIOUS DIMENSIONS*

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Recent analytical and numerical solutions of above systems are reviewed. Discussed results include: a) exact construction of the supersymmetric vacua in two space-time dimensions, and b) precise numerical calculations of the coexisting, continuous and discrete, spectra in the four-dimensional system, together with the identification of dynamical supermultiplets and SUSY vacua. New construction of the gluinoless SO(9) singlet state, which is vastly different from the empty state, in the ten-dimensional model is also briefly summarized.

Keywords: matrix models; quantum mechanics; non-abelian.

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1. Introduction

Supersymmetric Yang-Mils quantum mechanics (SYMQM) results from the dimensional reduction of the corresponding field theory to a single point in the D-1 dimensional space. Equivalently, it can be thought of as the effective quantum mechanics of zero momentum modes of the latter [1]. These systems were first considered in 80's [2] as simple models with supersymmetry [3]. Independently, zero-volume field theories (especially pure Yang-Mills) were being used as the starting point of the small volume expansion - an important theoretical development complementary to early lattice calculations [4,5,6]. In late 90's the models attracted a new wave of interest after the remarkable hypothesis of the equivalence, between the $D = 10, SU(\infty)$ SYMQM and M-theory of D0 branes, has been formulated [7,8,9,10,11,12].

We review some recent progress in studying the SU(2) models in two, four and ten space-time dimensions.

2. Exact supersymmetric vacuum in two space-time dimensions

The system, reduced from $D = 2$ to one (time) dimension [2], is described by the three real bosonic variables $x_a(t)$ and three complex, fermionic degrees of freedom $\psi_a(t)$, both in the adjoint representation of SU(2) , $a = 1, 2, 3$.

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The Hamiltonian reads

$$H = \frac{1}{2} p_a p_a + i g \epsilon_{abc} \psi_a^\dagger x_b \psi_c = \frac{1}{2} p_a p_a + g x_a G_a, \tag{1}$$

where the quantum operators $\psi, \psi^\dagger, x, p$ can be written in terms of the creation and annihilation operators

$$x_a = \frac{1}{\sqrt{2}}(a_a + a_a^\dagger), \;\; p_a = \frac{1}{i\sqrt{2}}(a_a - a_a^\dagger), \;\; \psi_a = f_a, \;\; \psi_a^\dagger = f_a^\dagger, \tag{2}$$

and gauge generators G_a read

$$G_a = \epsilon_{abc}(x_b p_c - i\psi_b^\dagger \psi_c). \tag{3}$$

The physical Hilbert space consists only of gauge invariant states. This constraint is accommodated by constructing all possible invariant combinations of creation operators (creators), and using them to generate a complete gauge invariant basis of states. There are four lower order creators:

$$(aa) \equiv a_a^\dagger a_a^\dagger, \;\; (af) \equiv a_a^\dagger f_a^\dagger, \;\; (aff) \equiv \epsilon_{abc} a_a^\dagger f_b^\dagger f_c^\dagger, \;\; (fff) \equiv \epsilon_{abc} f_a^\dagger f_b^\dagger f_c^\dagger. \tag{4}$$

In the physical basis the Hamiltonian (1) reduces to that of a free particle. Consequently the gauge invariant fermion number $F = f_a^\dagger f_a$ is conserved - the whole basis splits into the four sectors, each sector beginning with one of the following "base" states

$$|0_F\rangle = |0\rangle, \;\; |1_F\rangle = (af)|0\rangle, \;\; |2_F\rangle = (aff)|0\rangle, \;\; |3_F\rangle = (fff)|0\rangle. \tag{5}$$

To generate the whole basis in is then sufficient to act repeatedly on these four vectors with the bosonic creator (aa) [a]. The basis with a cutoff $N_{\rm cut}$ is then obtained by applying (aa) up to $N_{\rm cut}$ times. Obviously our cutoff is gauge invariant, since it is defined in terms of the gauge invariant creators.

In Refs. [13,14] such a basis was explicitly constructed in the "Mathematica representation" of quantum mechanics following by the numerical diagonalization of the Hamiltonian matrix for finite but large cutoffs. Resulting spectrum is indeed that of a free particle regularized in a harmonic oscillator basis. Eigenstates obtained in this way are in close correspondence with the gauge invariant plane waves of Ref. [2], $r^2 = x_a x_a$,

$$|0_F, k> = \frac{\sin(kr)}{kr}|+>, \quad |1_F, k> = \left(\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right)\frac{(x_a f_a^\dagger)}{r}|+>, \tag{6}$$

$$|3_F, k> = \frac{\sin(kr)}{kr}|->, \quad |2_F, k> = \left(\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right)\frac{(x_a f_a)}{r}|->. \tag{7}$$

Where $|\pm>$ were referred to as the "empty" and "filled" states. In the limit $k \to 0$ the second and fourth state vanish, which confirms that the empty (and filled) states are the (unpaired) supersymmetric vacua of the model. It is important to

[a]Notice that $(ff) = (af)^2 = (aff)^2 = (fff)^2 = 0$.

keep in mind that, e.g., the $|+>$ state *is not* empty in terms of the bosonic quanta. Obviously the $|0_F, 0_B>$ it is even not an eigenstate of H. On the other hand, we can easily construct the exact vacuum state(s) using results of the recursive approach of Ref. [14]. The Hamiltonian matrix in the $F=0$ basis is tridiagonal

$$< 2n|H|2n-2> = < 2n|H|2n-2> = -\frac{1}{4}\sqrt{2n+4n^2}, \quad < 2n|H|2n> = n+\frac{3}{4}, \tag{8}$$

and consequently the eigenequation $H|+> = 0|+>$ turns over into the recursive relation for the components of the vacuum state in the $F=0$ basis

$$\sqrt{(2n+2)(2n+3)}c_{2n+2} - (4n+3)c_{2n} + \sqrt{2n(2n+1)}c_{2n-2} = 0, \tag{9}$$

which determine the vacuum state

$$|+> = \Sigma_{n=0}^{\infty} c_{2n}|0_F, 2n>, \tag{10}$$

up to a normalization. Due to the particle-hole symmetry, similar equation holds for the $|->$ state in the 3_F sector.

Moreover, given the explicit representation of the supersymmetry generator Q in the $F=0$ basis [14], one readily verifies that it annihilates the above vacuum state

$$Q_{mn}c_n = 0 \tag{11}$$

as it should. Therefore the "empty" and "filled" qualifiers in Ref. [2] refer to the fermionic occupation numbers only. In terms of the bosonic quanta both supersymmetric vacua have the nontrivial structure given by Eq.(9).

3. Dynamical supermultiplets in four space-time dimensions

Four-dimensional system is of course much more rich and correspondingly more interesting. No analytical solutions are known in spite of various attempts [2]. There are now 15 degrees of freedom which upon quantization are represented by nine bosonic $a_b^{i\dagger}$ and six fermionic $f_a^{m\dagger}$ creation operators (see Ref. [13] for the details of the construction). Fermion number is again conserved which allowed separate diagonalization in each of seven $F = 0, 1, .., 6$ fermionic sectors. The gauge and rotationally invariant cutoff is again introduced as the maximal number of all bosonic quanta B_{max}. Direct implementation of the cut basis was practical only for $B_{max} \le 8$ and resulted in several thousands of all basis vectors. It was sufficient to perform the satisfactory comparison with earlier results known in the $F=0$ sector [5] which is identical with the pure Yang-Mills system. Present results are obtained for much higher cutoffs, in the range of $30-60$, in all fermionic sectors [15]. This progress was achieved by taking full advantage of rotational symmetry together with the generalization of the recursive construction of matrix elements, developed originally for the D=2 model [14]. Second dedicated method employs the separation of variables done first by Savvidy [16] and applied to the present system by van Baal [17]. It allows to reach the highest cutoffs up to date, but was done only for the two channels: $F = 0, 2$, both with the lowest total angular momentum $J = 0$. On the contrary

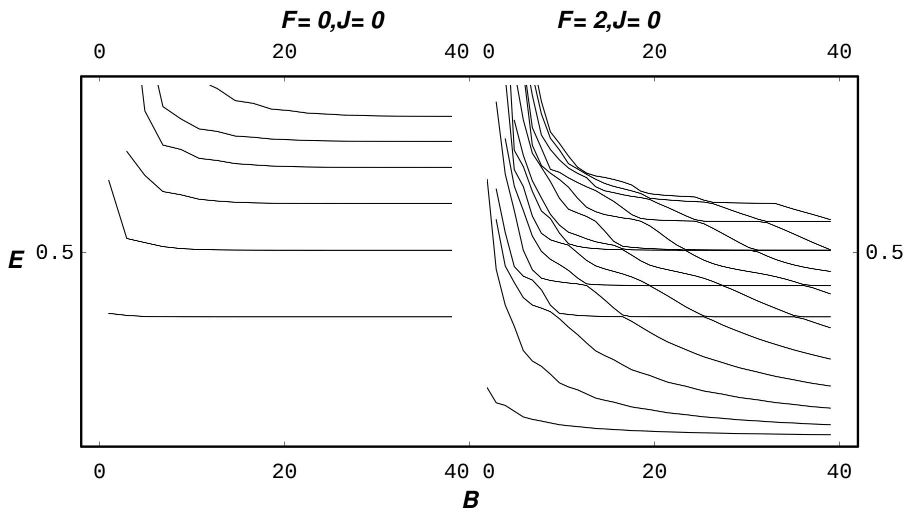

Fig. 1. Cutoff dependence of the spectrum in the zero- and two-gluino sectors.

the former approach works for all F's and J's. This gives now not only eigenenergies and eigenstates, but allows for the precise identification of supermultiplets and complete classification of the whole spectrum.

Fig. 1 shows the cutoff dependence of the spectrum in the $F = 0, J = 0$ and $F = 2, J = 0$ channels, obtained by separation of variables. One clearly sees the two families of states which differ dramatically by their convergence with the cutoff. Quickly convergent levels correspond to the discrete spectrum of localized states, while the slowly falling ones are the non-localized, scattering states which in the infinite cutoff limit form a continuous spectrum. This result nicely confirms the general expectations for the behavior of supersymmetric systems with the flat valley potentials . The continuous spectrum exists only in the "fermion rich" sectors supporting the general argument based on the fermion-boson cancellation of the zero-point energy. In the $F = 2$ sector both the discrete and continuous spectra coexist at the same energy, which is unusual but not excluded [18]. Moreover, the mixing between the discrete states and the continuum vanishes at infinite cutoff, rendering former stable as required by supersymmetry. Actual energies of the scattering states are encoded in the rate of fall with the cutoff seen in the Figure. They can be readily recovered in the scaling limit where the principal quantum number n varies with the cutoff B as $n \sim \sqrt{B}$ [19].

Similar results are now available in all channels of angular momentum J and fermion number F [15] [b]. This allowed rather detailed study of the supersymmetry

[b] The cutoffs reached with this approach have been recently pushed yet further and almost match precision obtained by separating variables [20].

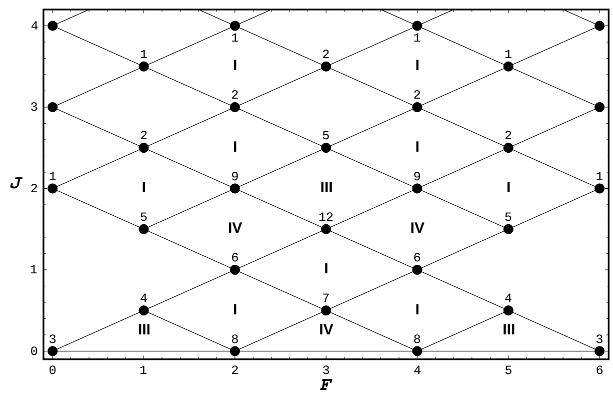

Fig. 2. Sizes of bases vs. numbers of associated supermultiplets for $B_{max} = 3$

induced relations among eigenstates and complete classification of the spectrum into dynamical supermultiplets. The problem is already seen in Fig. 1: the two lowest energies in the $F = 0, J = 0$ channel, $(0_F, 0_J)$, clearly agree with the first and third discrete levels of $(2_F, 0_J)$. Naturally one suspects the supersymmetry to cause this degeneracy. However the second level of $(2_F, 0_J)$ does not have a counterpart in the $(0_F, 0_J)$ channel. Therefore its superpartner must lie in yet another channel.

To illustrate the systematic study of dynamical supermultiplets it is helpful to consult a map of the Hilbert space shown in the F, J plane, Fig. 2. Dots represent specific (F, J) channels, while the adjacent Arabic numbers give sizes of bases which can be constructed at given cutoff (here $B_{max} = 3$). Notice that, equivalently, one could redistribute these "occupation numbers" into diamonds attached to each vertex. In this way we have generated the "occupation numbers" of diamonds denoted by roman numbers in the Figure. Each state can belong only to one diamond, therefore we have an array of local relations

$$d_i = \Sigma_{I|i} R_I, \tag{12}$$

where R_I denotes a multiplicity of a diamond I, d_i is a number of SO(3) multiplets in a channel i, and summation runs over I's adjacent to i. Eqs.(12) provide global constraints on bases in all channels. This situation holds for any odd cutoff B_{max}. The number of new states created with higher cutoff is such that they always fill the integer number of diamonds. This regularity is the necessary condition for supersymmetry. In fact above diamonds are nothing but supermultiplets with identical eigenenergies of all four members of the supermultiplet. In another words: while dots in the Figure are useful for labeling basis states, the diamonds are suitable for classifying eigenstates of the Hamiltonian.

To see that each diamond of Fig. 2 indeed represents a supermultiplet consider

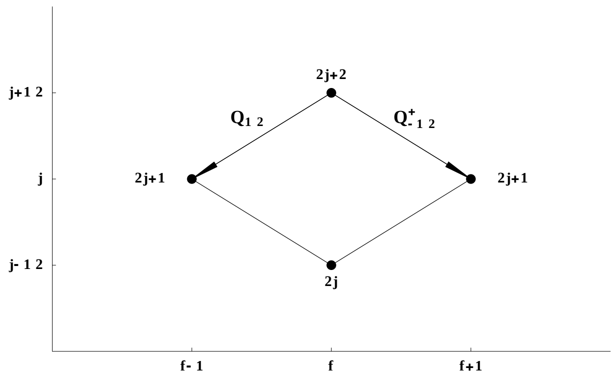

Fig. 3. A single massive supermultiplet described in the text. An example of the action of two supersymmetry generators is also shown.

the algebra of Weyl generators Q_m and $Q_m^\dagger$

$$\{Q_m, Q_n^\dagger\} = 2\delta_{mn} H, \ \ \{Q_m, Q_n\} = \{Q_m^\dagger, Q_n^\dagger\} = 0, \ \ m = \pm\frac{1}{2}. \tag{13}$$

Detailed form of these observables is not important here (see Ref. [15] for the details). It suffices to know that $Q_m^\dagger$ (Q_m) creates (annihilates) one fermion with the angular momentum $J_z = m$. Eq.(13) implies that these generators, when acting on the eigenstates of the Hamiltonian, can only move them within one diamond, see Fig. 3 . Similarly, repeated actions of up to four Q's transform eigenstates within the same supermultiplet. Further, for each eigenstate in the spectrum there exists a unique pair of generators which separately annihilate this state. That pair determines to which of the four neighboring supermultiplets the state belongs.

It was reassuring to observe how these relations are satisfied by our eigenstates. To this end we have constructed the *supersymmetry fractions*

$$q(j', i'|j, i) \equiv \frac{1}{4E_{j,i}} \left|\langle j'; i' \| Q^\dagger \| j; i\rangle\right|^2, \tag{14}$$

which analyze the supersymmetric image of an eigenstate $|F, j; i >$ in another channel $(F+1, j')$. They do not vanish only if the initial and final states belong to the same supermultiplet. Since the supersymmetry is restored only at the infinite cutoff, susy fractions have some residual cutoff dependence. It is however already weak for currently available values of B_{max} and for a large number of lower and intermediate states. Nevertheless monitoring this dependence is an important part of the whole analysis.

Combining together the spectrum and susy fractions allowed us to identify about a dozen of supermultiplets in a range of angular momenta. A sample of results is shown in Fig. 4. Very satisfactory restoration of supersymmetry in the lower part of the spectrum was found. It is also confirmed by our results on the Witten index which tends with the increasing cutoff, to the time independent constant of 1/4

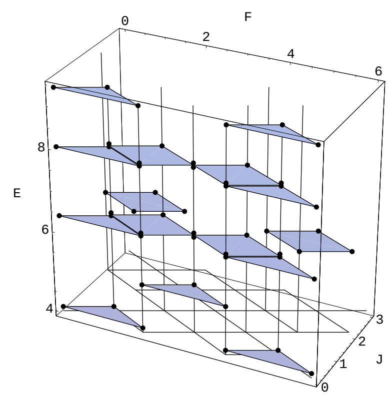

Fig. 4. A sample of dynamical supermultiplets identified for different gluino numbers F and angular momenta J.

[21,22]. Interestingly the splitting *between* some of the supermultiplets is very small and has been resolved only with the recent very precise data [20]. Apparently the system is rich enough that the energy differences between adjacent levels can vary by 2-3 orders of magnitude.

4. Summary

Supersymmetric Yang-Mills quantum mechanics is a good laboratory for modern model building. In the simplest, two space-time dimensional case, we have augmented well known, exact solution with the explicit construction of the supersymmetric vacuum state. Also recently, an exact restricted Witten index has been derived together with its manifestly supersymmetric spectral representation.

In four space-time dimensions a complete spectrum in various channels of fermionic number and angular momentum is now available. Expected, since a long time, coexistence of the localized and non-localized states has been explicitly confirmed. All eigenstates fall into well identifiable supermultiplets providing the dynamical realization of supersymmetry. There are two supersymmetric vacua in this model. They belong to the continuous spectrum of the scattering states extending into flat valleys of the potential. Not surprisingly their angular momentum $J = 0$. They are related by the particle-hole symmetry and have non-vanishing fermion numbers $F = 2$ and 4 respectively. An interesting pattern has been found among the scattering states. They exist only in the "central", c.f. Fig. 2, supermultiplets with $F = 2, 3, 4$ and only with *even* angular momentum of bosonic states. Some supermultiplets are almost, but not exactly, degenerate prompting new interesting questions.

Due to the lack of space we only signal recent developments in the $D = 10$ model [23]. In addition to the obvious combinatorial complexity (the system has 51 degrees of freedom), simultaneous Majorana-Weyl conditions for supersymmetric

gluinos result in an unusual situation where the fermionic number is not conserved, not only by the Hamiltonian, but also by Spin(9) rotations. In particular, in contradistinction to lower dimensions, the usual empty state is not rotationally invariant. It turns out that the simplest (no bosons) singlet state is in the most complex sector with twelve fermions. This state has recently been explicitly constructed and can now be used as the root for the subsequent building of the whole basis.

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References

1. L. Brink, J. H. Schwarz and J. Scherk, Nucl. Phys. **B121** (1977) 77.
2. M. Claudson and M. B. Halpern, Nucl. Phys. **B250** (1985) 689.
3. E. Witten, Nucl. Phys. **B188** (1981) 513.
4. M. Lüscher, Nucl. Phys. **B219** (1983) 233.
5. M. Lüscher and G. Münster, Nucl. Phys. **B232** (1984) 445.
6. P. van Baal, Acta Phys. Polon. **B20** (1989) 295.
7. T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys.Rev. **D55** (1997) 5112.
8. D. Bigatti, and L. Susskind, in Cargese 1997, *Strings, branes and dualities*, pp. 277-318, hep-th/9712072.
9. M. Porrati and A. Rosenberg, Nucl. Phys. **B515** (1998) 184, hep-th/9708119.
10. K. Becker and M. Becker, Nucl. Phys. **B506** (1997) 48, hep-th/9705091.
11. W. Taylor, Rev. Mod. Phys. **73** (2001) 419, hep-th/0101126.
12. M. B. Halpern and C. Schwartz, Int. J. Mod. Phys. **A13** (1998) 4367, hep-th/9712133.
13. J. Wosiek, Nucl. Phys. **B644** (2002) 85, hep-th/0203116; J. Kotanski and J. Wosiek, Nucl. Phys.**B** (Proc.Suppl.)**119** (2003) 932,hep-lat/0208067.
14. M. Campostrini and J. Wosiek, Phys. Lett. **B550** (2002) 121, hep-th/0209140.
15. M. Campostrini and J. Wosiek, hep-th/0407021.
16. G. K. Savvidy, Phys. Lett. **B 159** (1985) 325.
17. P. van Baal, *The Witten Index Beyond the Adiabatic Approximation*, in: Michael Marinov Memorial Volume, Multiple Facets of Quantization and Supersymmetry, eds. M. Olshanetsky and A. Vainshtein (World Scientific, Singapore, 2002), pp.556-584, hep-th/0112072; www.lorentz.leidenuniv.nl/vanbaal/susyYM.
18. M. Reeds and B. Simon, *Methods of modern mathematical physics, v.IV*, AP, 1987.
19. M. Trzetrzelewski and J. Wosiek, Acta Phys. Polon.**B35**(2004)1615, hep-th/0308007.
20. M. Campostrini, talk at the XII workshop on Statistical Mechanics and Field Theory, Bari, September 2004.
21. S. Sethi and M. Stern, Comm. Math. Phys., **194** (1998) 675, hep-th/9705046. P. Yi, Nucl. Phys. **B505** (1997) 307, hep-th/9704098.
22. A. V. Smilga, Nucl. Phys. **B266** (1986) 45.
23. J. Wosiek, to be published.